

STABILITY IN THE FAMILY OF ω -LIMIT SETS OF ALTERNATING SYSTEMS

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ABSTRACT. Let f and g be elements of $C(I)$ with $x \in I = [0, 1]$. We study the ω -limit sets $\omega(x, [f, g])$ generated by alternating trajectories of the form $\gamma(x, [f, g]) = \{x, f(x), g(f(x)), f(g(f(x))), \dots\}$, as well as the sets $\Lambda([f, g]) = \cup_{x \in I} \omega(x, [f, g])$ and $\mathcal{L}([f, g]) = \{\omega(x, [f, g]) : x \in I\}$. In particular, we show that

- (1) If g is constant on no interval $J \subseteq I$, then there exists a residual set $\mathcal{S} \subseteq C(I)$ so that the maps $\Lambda : C(I) \times C(I) \rightarrow \mathcal{K}$ and $\mathcal{L} : C(I) \times C(I) \rightarrow \mathcal{K}^*$ taking (f, g) to $\Lambda([f, g])$ and $\mathcal{L}([f, g])$, respectively, are both continuous at (f, g) whenever $f \in \mathcal{S}$.
- (2) The map $\omega : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ taking (x, f, g) to $\omega(x, [f, g])$ is in the second class of Baire, and for any $g \in C(I)$ there exists a residual set $\mathcal{T} \subseteq I \times C(I)$ so that ω is continuous at (x, f, g) whenever $(x, f) \in \mathcal{T}$.
- (3) If f is constant on no interval $J \subseteq I$, then there exists a residual set $\mathcal{D} \subseteq I \times C(I)$ so that $\omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$, where both $\omega(x, g \circ f)$ and $\omega(f(x), f \circ g)$ are adding machines of type ∞ , whenever $(x, g) \in \mathcal{D}$.

1. INTRODUCTION

The structure and stability of ω -limit sets are fundamental to our understanding of discrete dynamical systems. The results of [3], [8], [9] collectively provide not only complete characterizations of the structure of ω -limit sets for autonomous systems on $I = [0, 1]$, but also indicate how the structure of those ω -limit sets reflects the chaotic nature of the generating function. The line of inquiry found in [6], [20], [21] addresses the stability of ω -limit sets, and how the sets $\Lambda(f) = \cup_{x \in I} \omega(x, f)$ and $\mathcal{L}(f) = \{\omega(x, f) : x \in I\}$ are affected by slight perturbations in the generating function f . Taking advantage of the completeness of I , $C(I)$ and the Hausdorff metric space $(\mathcal{K}, \mathcal{H})$ of compact subsets of I , [12], [13], [24] and [20] describe what is typical for an ω -limit set.

Our goal is to extend our understanding for autonomous systems to non-autonomous systems. A lot of attention recently has been devoted to the study of non-autonomous discrete dynamical systems, that is, iterated systems of difference equations where the model changes with the discrete time. We recall, among others, the works of Alsharawi, Angelos, Cánovas, Cushing, D'Aniello and Oliveira, D'Aniello and Steele, Elaydi, Henson, Linero, Rakesh and Sacker ([1], [2], [10], [11], [14], [15], [17], [18] and [19]).

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Here is an overview of some of our main results. Let $I = [0, 1]$, the unit interval of the real line. By $C(I)$ we denote the space of all continuous maps from I to I . For $x \in I$, $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x , that is $B_\epsilon(x) = \{y \in I : |x - y| < \epsilon\}$. For f and h in $C(I)$, we write $\|h - f\| = d(f, h) = \sup_{x \in I} |f(x) - h(x)|$ and, hence, $B_\epsilon(f) = \{g \in C(I) : \|f - g\| < \epsilon\}$. Given a point x in I , the sequence $\gamma(x, [f, g]) = \{x, f(x), g(f(x)), f(g(f(x))), \dots\} = \gamma(x, g \circ f) \cup \gamma(f(x), f \circ g)$ is the trajectory of x for the continuous alternating system $[f, g]$ and the collection of all limit points of subsequences of the trajectory, denoted by $\omega(x, [f, g])$, is the ω -limit set of x for $[f, g]$. From [15], we know that $\omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g) = \omega(x, g \circ f) \cup f(\omega(x, g \circ f))$ is either a closed nowhere dense set, or a union of finitely many non-degenerate closed intervals. Moreover, $\mathcal{L}([f, g]) = \{\omega(x, [f, g]) : x \in I\}$ is closed with respect to the Hausdorff metric. Since $\Lambda([f, g]) = \cup_{x \in I} \omega(x, [f, g])$ is closed in I , and $\mathcal{L}([f, g])$ is closed in the Hausdorff metric space $(\mathcal{K}, \mathcal{H})$, we are able to make good use of the Baire category theorem as we consider how $\Lambda([f, g])$ and $\mathcal{L}([f, g])$ are affected by perturbing (f, g) in $C(I) \times C(I)$. We find that the sensibility of $\Lambda([f, g])$ and $\mathcal{L}([f, g])$ to perturbations in (f, g) has much to do with the sensibility of $\Lambda(g \circ f)$ and $\mathcal{L}(g \circ f)$ to perturbations in $g \circ f$. Specifically, if the map taking h to $\Lambda(h)$ is continuous at $g \circ f$, then the map taking (h, k) to $\Lambda([h, k])$ is continuous at (f, g) . The analogous result holds for $\mathcal{L}([f, g])$. Moreover, each of the maps Λ and $\mathcal{L}$ is continuous on a residual subset of $C(I) \times C(I)$. Similar results hold for the map $\omega : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ taking (x, f, g) to $\omega(x, [f, g])$:

if the map taking (y, h) to $\omega(y, h)$ is continuous at $(x, g \circ f)$, then the map taking (y, h, k) to $\omega(y, [h, k])$ is continuous at (x, f, g) ;

the map taking (x, f, g) to $\omega(x, [f, g])$ is a Baire 2 function continuous on a residual subset of $I \times C(I) \times C(I)$, and

at a typical point (x, f, g) , $\omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$, where both $\omega(x, g \circ f)$ and $\omega(f(x), f \circ g)$ are adding machines of type ∞ .

Here is how we proceed. Sections 2 and 3 are dedicated to periodic, recurrent and chain recurrent sets of continuous alternating systems. Basic properties of these sets are developed, and we see that some, but certainly not all, of the properties associated with $P(f)$, $R(f)$ and $CR(f)$ are maintained in the new setting. Section 4 is dedicated to the study of $\Lambda : C(I) \times C(I) \rightarrow \mathcal{K}$ given by $(f, g) \mapsto \Lambda([f, g])$ and Section 5 considers the map $\mathcal{L} : C(I) \times C(I) \rightarrow \mathcal{K}^*$ given by $(f, g) \mapsto \mathcal{L}([f, g])$. Main results are found in Theorem 4.8, Theorem 4.13, Theorem 5.3, Theorem 5.4 and Corollary 5.15. In Section 6 we study the continuity structure of the map $\omega : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ given by $(x, f, g) \mapsto \omega(x, [f, g])$, while Section 7 describes what is typical for $\omega(x, [f, g])$. Main results are found in Theorem 6.1, Theorem 6.2, Corollary 6.5 and Corollary 7.7.

2. PERIODIC AND RECURRENT POINTS OF $[f, g]$

Definition 2.1. Suppose f and g are in $C(I)$. We let $\tilde{P}([f, g]) = \{x : x \in \omega(x, [f, g]) \text{ and } \omega(x, [f, g]) \text{ is finite}\}$.

Remark 2.2. Suppose $x \in \tilde{P}([f, g])$. Then $\omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$ is finite. Since x is contained in at least one of the periodic orbits $\omega(x, g \circ f)$ or $\omega(f(x), f \circ g)$, it follows that $\tilde{P}([f, g]) \subseteq P(g \circ f) \cup P(f \circ g)$. If x is periodic under $g \circ f$, then $x \in \omega(x, [f, g])$ and $\omega(x, [f, g])$ is finite. Therefore $P(g \circ f) \subseteq \tilde{P}([f, g])$. If $x \in P(f \circ g)$ then $\omega(x, f \circ g) \cup \omega(g(x), g \circ f) = \omega(g(x), [f, g])$ so that $g(x) \in \tilde{P}([f, g])$. Thus, $g(P(f \circ g)) \subseteq \tilde{P}([f, g])$. Suppose $x \in P(g \circ f)$. Then $x \in \omega(x, g \circ f)$ finite, so that $f(x) \in \omega(f(x), f \circ g)$ finite. Thus, $f(P(g \circ f)) \subseteq P(f \circ g)$. Now, $(g \circ f)(P(g \circ f)) = g(P(f \circ g)) \subseteq P(g \circ f)$, and $(g \circ f)(P(g \circ f)) = P(g \circ f)$.

We summarize the previous remark in the following lemma.

Lemma 2.3. Let f and g in $C(I)$. Then

- (1) $\tilde{P}([f, g]) \subseteq P(g \circ f) \cup P(f \circ g)$,
- (2) $P(g \circ f) \subseteq \tilde{P}([f, g])$,
- (3) $g(P(f \circ g)) \subseteq \tilde{P}([f, g])$, and
- (4) $f(P(g \circ f)) = P(f \circ g)$

Definition 2.4. Suppose f and g are in $C(I)$. We let $R([f, g]) = \{x \in I : x \in \omega(x, [f, g])\}$.

Lemma 2.5. Suppose f and g are in $C(I)$. Then, $f(R(g \circ f)) = R(f \circ g)$.

Proof. Let $x \in R(g \circ f)$. Then $x \in \omega(x, g \circ f)$ so that $f(x) \in f(\omega(x, g \circ f)) = \omega(f(x), f \circ g)$. Thus, $f(x) \in R(f \circ g)$ and $f(R(g \circ f)) \subseteq R(f \circ g)$. Now, by twice applying what was just demonstrated, we obtain

$$(f \circ g)(R(f \circ g)) = f(g(R(f \circ g))) \subseteq f(R(g \circ f)) \subseteq R(f \circ g).$$

Since $f \circ g \in C(I)$, it follows that $(f \circ g)(R(f \circ g)) = R(f \circ g)$, so we have

$$(f \circ g)(R(f \circ g)) = f(g(R(f \circ g))) \subseteq f(R(g \circ f)) \subseteq R(f \circ g) = (f \circ g)(R(f \circ g)).$$

Hence the thesis: $f(R(g \circ f)) = R(f \circ g)$. $\square$

Remark 2.6. By exchanging f and g we also have $g(R(f \circ g)) = R(g \circ f)$.

From the proof of the previous lemma we deduce the following result:

Lemma 2.7. Suppose f and g are in $C(I)$. Then, $R(g \circ f) \subseteq R([f, g])$.

Proof. If $x \in R(g \circ f)$, then $x \in \omega(x, g \circ f)$ so that $x \in \omega(x, [f, g])$. Thus, $x \in R([f, g])$. $\square$

Lemma 2.8. Suppose f and g are in $C(I)$. If $y \in R([f, g])$, then $y \in \omega(y, g \circ f)$ or $g(y) \in \omega(y, g \circ f)$.

Proof. If $y \in R([f, g])$ then $y \in \omega(y, g \circ f) \cup \omega(f(y), f \circ g)$. In case 1, $y \in \omega(y, g \circ f)$. In case 2, $y \in \omega(f(y), f \circ g)$. So, we have

$$g(y) \in \omega((g \circ f)(y), g \circ f) = \omega(y, g \circ f).$$

$\square$

Corollary 2.9. Suppose f and g are in $C(I)$. If $y \in \tilde{P}([f, g])$, then $y \in P(g \circ f)$ or $g(y) \in P(g \circ f)$.

Example 2.10. Here is an example of $x \in P([f, g]) \subseteq R([f, g])$ so that

- (1) $x \in \omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$ (this follows by definition),
- (2) $x \in \omega(f(x), f \circ g)$, and
- (3) $x \notin \omega(x, g \circ f)$.

Let $x = \frac{1}{2}$. Fix $z \in I \setminus \{\frac{1}{3}, \frac{1}{2}\}$. Let $f, g : I \rightarrow I$ be surjective continuous maps such that

$$f(\frac{1}{3}) = \frac{1}{2}, f(\frac{1}{2}) = z,$$

and

$$g(z) = g(\frac{1}{2}) = \frac{1}{3}.$$

This gives

$$(g \circ f)(\frac{1}{2}) = \frac{1}{3}, (g \circ f)(\frac{1}{3}) = \frac{1}{3}$$

and

$$(f \circ g)(z) = \frac{1}{2}, (f \circ g)(\frac{1}{2}) = \frac{1}{2}.$$

Then

$$\begin{aligned} \omega(\frac{1}{2}, [f, g]) &= \omega(\frac{1}{2}, g \circ f) \cup \omega(f(\frac{1}{2}), f \circ g) \\ &= \omega(\frac{1}{2}, g \circ f) \cup \omega(z, f \circ g) \\ &= \{\frac{1}{3}\} \cup \{\frac{1}{2}\}. \end{aligned}$$

We next develop the adding machines. As in [13] and [12], much of the terminology is borrowed from [5]. Let $\alpha = (\alpha(1), \alpha(2), \dots, \alpha(n), \dots) \in (\mathbb{N} \setminus \{1\})^{\mathbb{N}}$, and set

$$\Delta_\alpha = \prod_{i=1}^{\infty} \mathbb{Z}_{\alpha(i)},$$

where $\mathbb{Z}_k = \{0, \dots, k-1\}$. We use the product topology on Δ_α . Hence, as a topological space, it is homeomorphic to the Cantor space. Instead of the usual coordinate-wise addition, we add two elements of Δ_α with “carry over” to the right. More precisely, if $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ are in Δ_α , then

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (z_1, z_2, \dots),$$

where $z_1 = x_1 + y_1 \bmod (\alpha(1))$ and, in general, z_i is defined recursively as $z_i = x_i + y_i + \epsilon_{i-1} \bmod (\alpha(i))$ where $\epsilon_{i-1} = 0$ if $x_{i-1} + y_{i-1} + \epsilon_{i-2} < \alpha(i-1)$ and $\epsilon_{i-1} = 1$ otherwise. If we let f_α be the “+1” map, that is $f_\alpha(x_1, x_2, \dots) = (x_1, x_2, \dots) + (1, 0, 0, \dots)$, then $(\Delta_\alpha, f_\alpha)$ is a dynamical system known in various contexts as a solenoid, adding machine or odometer. We refer to f_α as an adding machine or an odometer ([4], [5]).

Fix $\alpha \in (\mathbb{N} \setminus \{1\})^{\mathbb{N}}$. We define a function M_α from the set of primes into $\mathbb{N} \cup \{\infty\}$ as follows. For each prime p , let

$$M_\alpha(p) = \sum_{i=1}^{\infty} n(i),$$

where $n(i)$ is the largest power of p which divides $\alpha(i)$. We call *odometers of type ∞* the odometers associated with those α for which $M_\alpha(p) = \infty$ for all p . Theorems 2.11 and 2.12 give a beautiful characterization of adding machines up to topological conjugacy is due to Block and Keesling ([5]).

Theorem 2.11. *Let $\alpha \in (\mathbb{N} \setminus \{1\})^\mathbb{N}$. Let $m_i = \alpha(1)\alpha(2)\dots\alpha(i)$ for each i . Let $f : X \rightarrow X$ be a continuous map of a compact topological space X . Then f is topologically conjugate to f_α if and only if (1), (2), and (3) hold.*

- (1) *For each positive integer i , there is a cover P_i of X consisting of m_i pairwise disjoint, nonempty, clopen sets which are cyclically permuted by f .*
- (2) *For each positive integer i , P_{i+1} partitions P_i .*
- (3) *If $W_1 \supset W_2 \supset W_3 \supset \dots$ is a nested sequence with $W_i \in P_i$ for each i , then $\bigcap_{i=1}^\infty W_i$ consists of a single point.*

Moreover, in this case statement (4) also holds.

- (4) *X is metrizable and if $\text{mesh}(P_i)$ denotes the maximum diameter of an element of the cover P_i , then $\text{mesh}(P_i) \rightarrow 0$ as $i \rightarrow \infty$.*

Theorem 2.12. ([5]: Corollary 2.8) *Let $\alpha, \beta \in (\mathbb{N} \setminus \{1\})^\mathbb{N}$. Then f_α and f_β are topologically conjugate if and only if $M_\alpha = M_\beta$.*

Example 2.13. *If $f \in C(I)$, then $\overline{P(f)} = \overline{R(f)}$ is called the center of f [4]. The following example shows that $\tilde{P}([f, g])$ can be a proper subset of $\overline{R([f, g])}$.*

- (1) *Take $x = \frac{1}{4}$. Let $l : I \rightarrow I$ be continuous so that*

- a. $\omega(\frac{3}{4}, l)$ is an odometer of type ∞ ,
- b. $\frac{1}{4} \in \omega(\frac{3}{4}, l)$,
- c. $\frac{3}{4} \notin \omega(\frac{3}{4}, l)$.

- (2) *Take $f : I \rightarrow I$ a homeomorphism so that $f(\frac{1}{4}) = \frac{3}{4}$.*

- (3) *Set $g = f^{-1} \circ l$.*

- (4) $\omega(f(x), f \circ g) = \omega(\frac{3}{4}, f \circ f^{-1} \circ l) = \omega(\frac{3}{4}, l)$ is ∞ -adic.

- (5) *Since*

$$y \in \omega(f(z), f \circ g) \Rightarrow g(y) \in \omega(g(f(z)), g \circ f) = \omega(z, g \circ f),$$

it follows that

$$\frac{1}{4} \notin \omega(\frac{1}{4}, g \circ f)$$

as

$$\frac{1}{4} \notin \omega(\frac{1}{4}, g \circ f) \Leftrightarrow \frac{1}{4} \notin g(\omega(\frac{3}{4}, f \circ g)) = (f^{-1} \circ l)(\omega(\frac{3}{4}, l)) = f^{-1}(\omega(\frac{3}{4}, l))$$

and

$$\frac{3}{4} \notin \omega(\frac{3}{4}, l).$$

- (6) *Since $\frac{3}{4} \notin \omega(\frac{3}{4}, l) = \omega(\frac{3}{4}, f \circ g)$ we can modify g in a small neighborhood of $\frac{3}{4}$, and not affect the dynamics on $\omega(\frac{3}{4}, f \circ g)$:*

- a. *Let $\epsilon > 0$ so that $B_\epsilon(\frac{3}{4}) \cap \omega(f(\frac{1}{4}), f \circ g) = \emptyset$.*
- b. *Define $g_1 : I \rightarrow I$ so that*
 - i. $g_1 = g$ on $I \setminus B_\epsilon(\frac{3}{4})$
 - ii. $g_1(\frac{3}{4}) \in \omega(\frac{1}{4}, g \circ f)$

- iii. extend g_1 to all of I (and, hence, on all of $B_\epsilon(\frac{3}{4})$) using the Tietze extension theorem.

Now, $g_1(f(\frac{1}{4})) \in \omega(\frac{1}{4}, g_1 \circ f) = \omega(\frac{1}{4}, g \circ f)$.

- c. Let $\epsilon > 0$ so that $B_\epsilon(\frac{1}{4}) \cap \omega(\frac{1}{4}, g_1 \circ f) = \emptyset$. Define $f_1 : I \rightarrow I$ so that
 - i. $f_1 = f$ on $I \setminus B_\epsilon(\frac{1}{4})$
 - ii. $f_1(z) = f(\frac{1}{4})$ for every $z \in \overline{B_\epsilon(\frac{1}{4})}$
 - iii. extend f_1 to all of I (and, hence, on all of $B_\epsilon(\frac{1}{4})$) using the Tietze extension theorem.

(7) Then

- a. $\frac{1}{4} \in \omega(\frac{1}{4}, [f_1, g_1]) = \omega(\frac{1}{4}, [f, g]) = \omega(\frac{1}{4}, g \circ f) \cup \omega(\frac{3}{4}, f \circ g)$, as $f_1(\frac{1}{4}) = f(\frac{1}{4}) = \frac{3}{4}$.
- b. $\frac{1}{4} \in \omega(f_1(\frac{1}{4}), f_1 \circ g_1) = \omega(\frac{3}{4}, l)$.
- c. $\frac{3}{4} \notin \omega(f_1(\frac{1}{4}), f_1 \circ g_1) = \omega(\frac{3}{4}, l)$ so that $\frac{1}{4} \notin \omega(\frac{1}{4}, g_1 \circ f_1) = f^{-1}(\omega(\frac{3}{4}, l))$

(8) $\frac{1}{4} \in \omega(\frac{1}{4}, [f_1, g_1]) = K$ implies $\frac{1}{4} \in R([f_1, g_1])$.

If $z \in B_\epsilon(\frac{1}{4})$, then $\omega(z, [f_1, g_1]) = K$, so that $B_\epsilon(\frac{1}{4}) \cap \tilde{P}([f_1, g_1]) = \emptyset$.

3. CHAIN RECURRENT SET OF $[f, g]$

Let $f \in C(I)$. We recall that the set $CR(f)$ of all *chain recurrent points* of f is defined by $x \notin CR(f)$ if there exists an open set U with $f(\overline{U}) \subseteq U$ such that $x \notin \overline{U}$, $f(x) \in U$ ([4]: page 112).

Let $\epsilon > 0$ be given and let x, y , be any points of I . An ϵ -chain, or *pseudo-orbit*, from x to y is a finite sequence $\{x_0, x_1, \dots, x_n\}$ of points of I with $x = x_0$, $y = x_n$ and $|f(x_{k-1}) - x_k| < \epsilon$ for $k = 1, \dots, n$ ([4]: page 115).

Proposition 3.1. ([4]: Corollary V.46) *Let $f \in C(I)$. A point $x \in I$ is chain recurrent if and only if, for every $\epsilon > 0$, there is an ϵ -chain from x to itself.*

Proposition 3.2. ([4]: Proposition V.37, Corollary V.40) *Let $f \in C(I)$. The chain recurrent set $CR(f)$ is closed and strongly invariant.*

Proposition 3.3. *Let $f \in C(I)$. Then $x \in CR(f)$ if and only if any open neighbourhood V of f in $C(I)$ contains a map g such that $x \in P(g)$.*

Proof. $\Rightarrow$ If $x \in CR(f)$ then any open neighbourhood V of f in $C(I)$ contains a map g such that $x \in P(g)$ ([4]: Lemma V.49).

$\Leftarrow$ Let $\epsilon > 0$. It suffices to show that there exists an ϵ -chain from x to itself. Let $g \in B_\epsilon(f)$ such that $x \in P(g)$. Say $x_0 = x$, $g^i(x) = x_i$ for every $0 \leq i \leq n$ and $g^{n+1}(x) = x$. That is, $\text{orb}(x, g) = \{x_0, x_1, \dots, x_n\} = \omega(x, g)$. Since, $g \in B_\epsilon(f)$, $|g(z) - f(z)| < \epsilon$ for all $z \in I$. In particular, $|f(x_{k-1}) - x_k| = |f(x_{k-1}) - g(x_{k-1})| < \epsilon$ for $k = 1, 2, \dots, n$. □

Definition 3.4. *Let f and g be elements of $C(I)$. A point $x \in I$ is a chain recurrent point of $[f, g]$, and we write $x \in CR([f, g])$, if every open neighbourhood V of (f, g) in $C(I) \times C(I)$ contains some (f_1, g_1) such that $x \in \tilde{P}([f_1, g_1])$.*

Lemma 3.5. *Let f and g be elements of $C(I)$. Then $CR(g \circ f) \subseteq CR([f, g])$.*

Proof. Let $x \in CR(g \circ f)$ with V an open neighbourhood of (f, g) . Let $\epsilon > 0$ so that $B_\epsilon(f \circ g) \subseteq V$. By Block and Coppel, Lemma V.49, there exists $h \in B_\epsilon(I)$ so that $x \in P(h \circ g \circ f) = P((h \circ g) \circ f)$. Thus, $x \in \tilde{P}([f, h \circ g])$, and $(f, h \circ g) \in B_\epsilon(f, g)$. $\square$

Lemma 3.6. *Let f and g be elements of $C(I)$. Then $CR([f, g]) \subseteq CR(g \circ f) \cup CR(f \circ g)$.*

Proof. Let $x \in CR([f, g])$. Fix $n \in \mathbb{N}$ with $(f_n, g_n) \in B_{\frac{1}{n}}((f, g))$ such that $x \in \tilde{P}([f_n, g_n])$. Say $\omega(x, [f_n, g_n]) = \omega(x, g_n \circ f_n) \cup \omega(f_n(x), f_n \circ g_n) = A_n \cup B_n$, where A_n and B_n are necessarily periodic. Should $x \in B_n$, then $B_n = \omega(x, f_n \circ g_n)$. Since $(f_n, g_n) \rightarrow (f, g)$ as $n \rightarrow \infty$, and x is contained in at least one of the collections $\{\omega(x, g_n \circ f_n) : n \geq 1\}$, $\{\omega(x, f_n \circ g_n) : n \geq 1\}$ infinitely often, it follows that x is contained in at least one of the sets $CR(g \circ f)$, $CR(f \circ g)$. We conclude that $x \in CR(g \circ f) \cup CR(f \circ g)$. $\square$

Lemma 3.7. *Let f and g be elements of $C(I)$. Then $g(CR(f \circ g)) \subseteq CR(g \circ f) \subseteq CR([f, g])$.*

Proof. Let $x \in CR(f \circ g)$. Fix $\epsilon > 0$. Since $g \in C(I)$, g is uniformly continuous on I . Let $\delta > 0$ so that $|g(x) - g(y)| < \epsilon$ whenever $|x - y| < \delta$. Say $\{x_0, x_1, \dots, x_n\}$ is a δ -chain from x to itself for $f \circ g$. Then, $|(f \circ g)(x_{k-1}) - x_k| < \delta$ for $k = 1, 2, \dots, n$. Since $(g \circ f \circ g)(x) = (g \circ f)(g(x)) = g((f \circ g)(x))$ it follows that $|(g \circ f)(g(x_{k-1})) - g(x_k)| < \epsilon$ for $k = 1, \dots, n$, and $\{g(x_0), g(x_1), \dots, g(x_n)\}$ is an ϵ -chain from $g(x)$ to itself for $g \circ f$. Thus, $g(x) \in CR(g \circ f)$. Hence, $g(CR(f \circ g)) \subseteq CR(g \circ f)$. By Lemma 3.5,

$$g(CR(f \circ g)) \subseteq CR(g \circ f) \subseteq CR([f, g]).$$

$\square$

Lemma 3.8. *Let f and g be elements of $C(I)$. Then $g(CR(f \circ g)) = CR(g \circ f)$.*

Proof. By Lemma 3.7 $g(CR(f \circ g)) \subseteq CR([f, g])$. Moreover, $(f \circ g)(CR(f \circ g)) = f(g(CR(f \circ g))) \subseteq f(CR(g \circ f)) \subseteq CR(f \circ g)$. Couple this with the fact that $(f \circ g)(CR(f \circ g)) = CR(f \circ g)$, and conclude that $g(CR(f \circ g)) = CR(g \circ f)$. $\square$

4. ω -LIMIT POINTS OF $[f, g]$

Definition 4.1. ([21]) *Let $f \in C(I)$. If $x \in P(f)$ has period n , and if any neighborhood of x contains points u, v such that $f^n(u) < u$ and $f^n(v) > v$, then x is an essential periodic point. Let $S_0(f)$ represent the essential periodic points of f .*

Lemma 4.2. ([15]: Lemma 3.1) *If $[f, g]$ is a continuous alternating system on I and $x \in I$, then $\omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$.*

Lemma 4.3. ([15]: Lemma 3.2) *If f and g are in $C(I)$ and $x \in I$, then*

$$\omega(f(x), f \circ g) = f(\omega(x, g \circ f)).$$

Proposition 4.4. ([15]: Lemma 4.1) *Suppose f and g are in $C(I)$, and $f : \mathcal{L}(g \circ f) \rightarrow \mathcal{L}(f \circ g)$ such that $\sigma \mapsto f(\sigma)$. Then f is a bijection.*

We recall that $(\mathcal{K}, \mathcal{H})$ denotes the class of nonempty closed sets $\mathcal{K}$ in I endowed with the Hausdorff metric $\mathcal{H}$. It follows that $(\mathcal{K}, \mathcal{H})$ is compact [7].

Definition 4.5. Let $\Lambda : C(I) \times C(I) \rightarrow \mathcal{K}$ be given by $(f, g) \mapsto \Lambda([f, g])$, where $\Lambda([f, g]) = \cup_{x \in I} \omega(x, [f, g])$.

Remark 4.6. For every $(f, g) \in C(I) \times C(I)$, $\Lambda([f, g]) = \Lambda(g \circ f) \cup \Lambda(f \circ g)$, so that $\Lambda([f, g])$ is a closed subset of I . This follows immediately from Corollary IV.12 in [4], since both $\Lambda(g \circ f)$ and $\Lambda(f \circ g)$ are closed.

Theorem 4.7. ([23]) Let $f \in C(I)$. If $\overline{S_0(f)} = CR(f)$ then $\Lambda : C(I) \rightarrow \mathcal{K}$ is continuous at f .

Theorem 4.8. Let $(f, g) \in C(I) \times C(I)$. If $\overline{S_0(g \circ f)} = CR(g \circ f)$ then $\Lambda : C(I) \times C(I) \rightarrow \mathcal{K}$ is continuous at (f, g) .

Proof. Let $\overline{S_0(f \circ g)} = CR(g \circ f)$, and suppose that $\{(f_n, g_n)\}_{n \in \mathbb{N}} \subseteq C(I) \times C(I)$ so that $(f_n, g_n) \rightarrow (f, g)$. It suffices to show that $\Lambda([f_n, g_n])$ converges to $\Lambda([f, g])$ as n tends to ∞ . Since $\overline{S_0(g \circ f)} = CR(g \circ f)$ and (f_n, g_n) converges to (f, g) as n tends to ∞ , it follows that $g_n \circ f_n \rightarrow g \circ f$ and hence, by Theorem 4.7, $\Lambda(g_n \circ f_n) \rightarrow \Lambda(g \circ f)$. It remains to show that $\Lambda(f_n \circ g_n) \rightarrow \Lambda(f \circ g)$ in order to complete the proof, as $\Lambda([f, g]) = \Lambda(g \circ f) \cup \Lambda(f \circ g)$, by definition. Recall that $f_n(\Lambda(g_n \circ f_n)) = \Lambda(f_n \circ g_n)$ and that $f_n \rightarrow f$. Thus, $\Lambda(f_n \circ g_n) = f_n(\Lambda(g_n \circ f_n)) \rightarrow f(\Lambda(g \circ f)) = \Lambda(f \circ g)$. $\square$

Corollary 4.9. The map $\Lambda : C(I) \times C(I) \rightarrow \mathcal{K}$ is continuous at (f, g) if the map $\Lambda : C(I) \rightarrow \mathcal{K}$ is continuous at $g \circ f$.

Proof. This follows from Theorem 4.8 and Proposition 3.1 in [24]. $\square$

Lemma 4.10. (Banach) ([7]) There exists $\mathcal{I} \subseteq \mathcal{C}(I)$ so that $\mathcal{I}$ is residual, and if $f \in \mathcal{I}$, then f is constant on no subinterval of I .

Proposition 4.11. Let f and g be in $\mathcal{I}$. If $x \in CR(f \circ g)$ then any open neighborhood of f contains a function f_1 for which $x \in S_0(f_1 \circ g)$.

Proof. Assume first that x is an element of $P(f \circ g)$. Let V be a neighborhood of f , and take $\epsilon > 0$ so that $B_\epsilon(f) \subset V$. Since f is continuous on I , there exists $\delta > 0$ so that $|f(x) - f(y)| < \frac{\epsilon}{4}$ for all x, y in I which satisfy $|x - y| < \delta$. Suppose $x \in P(f \circ g)$ is of period n . Set $\omega(x, f \circ g) = \{x_0, x_1, \dots, x_{n-1}\}$, where $(f \circ g)(x_{i-1}) = x_i$, for $1 \leq i \leq n-2$ and $(f \circ g)(x_{n-1}) = x_0 = x$. Let $\gamma_1 = \min_{i \neq j} |x_i - x_j|$ and $\beta_1 = \min_{i \neq j} |g(x_i) - g(x_j)|$, and set $\gamma = \min\{\gamma_1, \frac{\epsilon}{4}, \delta\}$ with $\beta = \min\{\beta_1, \frac{\epsilon}{4}, \delta\}$. Since $(f \circ g)^n(x_i) = x_i$ with f and g in $\mathcal{I}$, we see that $g((f \circ g)^{n-1}(B_{\frac{\gamma}{4}}(x_i)))$ contains $g(x_{i-1})$ and an interval. Say $\langle g(x_{i-1}), z_{i-1} \rangle$ is contained in $g((f \circ g)^{n-1}(B_{\frac{\gamma}{4}}(x_i))) \cap B_{\frac{\beta}{8}}(g(x_{i-1}))$, where $\langle g(x_{i-1}), z_{i-1} \rangle$ is either $[g(x_{i-1}), z_{i-1}]$ or $[z_{i-1}, g(x_{i-1})]$, and $z_{i-1} \neq g(x_{i-1})$. Take $\{b_{i,k}\}_k \subseteq \langle g(x_{i-1}), z_{i-1} \rangle$ so that $b_{i,k} \rightarrow g(x_{i-1})$ as k tends to ∞ and is strictly monotone, and set, for every $k \in \mathbb{N}$, $l_{i,k} = \frac{b_{i,k} + b_{i,k+1}}{2}$. We now pull back the sequences $\{b_{i,k}\}_k$ and $\{l_{i,k}\}_k$ to their pre-images in $B_{\frac{\gamma}{4}}(x_i)$. In particular, let, for every $k \in \mathbb{N}$, $\tilde{b}_{i,k}$ be contained in $B_{\frac{\gamma}{4}}(x_i)$ so that $g((f \circ g)^{n-1}(\tilde{b}_{i,k})) = b_{i,k}$. If there is more than one possible choice for $\tilde{b}_{i,k}$, pick $\tilde{b}_{i,k}$ so that $|x_i - \tilde{b}_{i,k}|$ is minimal. Take $\tilde{l}_{i,k}$ in $B_{\frac{\gamma}{4}}(x_i)$ in a similar fashion, so that $g((f \circ g)^{n-1}(\tilde{l}_{i,k})) = l_{i,k}$. We now define the function f_1 :

- (1) $f_1 = f$ on $I \setminus \cup_{i=1}^n B_{\frac{\beta}{4}}(g(x_{i-1}))$,
- (2) $f_1(g(x_{i-1})) = f(g(x_{i-1})) = x_i$ for $i = 0, 1, \dots, n-1$, so that we maintain our periodic orbit

- (3) $f_1(b_{i,k}) = \min\{1, x_i + 2|x_i - \tilde{b}_{i,k}|\}$. Then
 a. $f_1(b_{i,k}) = f_1(g(f \circ g)^{n-1}(\tilde{b}_{i,k})) > \tilde{b}_{i,k}$
 b. $f_1(b_{i,k}) \rightarrow (f_1 \circ g)(x_{i-1}) = x_i$ as $k \rightarrow \infty$
- (4) $f_1(l_{i,k}) = \max\{0, x_i - 2|x_i - \tilde{l}_{i,k}|\}$. Then
 a. $f_1(l_{i,k}) = f_1(g(f \circ g)^{n-1}(\tilde{l}_{i,k})) < \tilde{l}_{i,k}$
 b. $f_1(l_{i,k}) \rightarrow (f_1 \circ g)(x_{i-1}) = x_i$ as $k \rightarrow \infty$
- (5) Extend f_1 to the remainder of $\cup_{i=1}^n B_{\frac{\epsilon}{4}}(g(x_{i-1}))$ using the Tietze extension theorem so that $d(f, f_1) < \epsilon$.

Now, let $x \in CR(f \circ g) \setminus P(f \circ g)$. Let V be a neighborhood of f , and take $\epsilon > 0$ so that $B_\epsilon(f) \subset V$. By Proposition 3.3, there exists h in $B_{\frac{\epsilon}{2}}(f)$ such that $x \in P(h \circ g)$. By the first part of the proof applied to the point x and the map h , there exists f_1 in $B_{\frac{\epsilon}{2}}(h)$ for which $x \in S_0(f_1 \circ g)$. As $\|f_1 - f\| \leq \|f_1 - h\| + \|h - f\| < \epsilon$, we have $f_1 \in B_\epsilon(f) \subset V$. $\square$

Remark 4.12. In Proposition 4.11 we have proved a bit more. We now have $orb(x, f_1 \circ g) \subseteq S_0(f_1 \circ g)$. So, for any $\sigma > 0$ there exists a neighborhood U containing f_1 so that $f_2 \in U$ implies $orb(\tilde{x}, f_2 \circ g) \subseteq S_0(f_2 \circ g)$, where $H(orb(\tilde{x}, f_2 \circ g), orb(x, f_1 \circ g)) < \sigma$.

Theorem 4.13. Let $g \in \mathcal{I}$. Then, there exists a residual set $\mathcal{S} \subseteq C(I)$ such that $\overline{S_0(f \circ g)} = CR(f \circ g)$ whenever $f \in \mathcal{S}$.

Proof. (1) Since $S_0(f \circ g) \subseteq CR(f \circ g)$ and $CR(f \circ g)$ is closed in I it follows that $\overline{S_0(f \circ g)} \subseteq CR(f \circ g)$. To show that $H(\overline{S_0(f \circ g)}, CR(f \circ g)) < \epsilon$, then, it suffices to show that for any $x \in CR(f \circ g)$ there exists $y \in S_0(f \circ g)$ so that $|x - y| < \epsilon$. Set $S_n = \{f \in \mathcal{I} : \mathcal{H}(\overline{S_0(f \circ g)}, CR(f \circ g)) < \frac{1}{n}\}$. Since $\mathcal{S} = \cap_{n=1}^\infty S_n$, we need to show that S_n is both dense and open in $\mathcal{I}$.

(2) We verify that S_n is a dense subset of $\mathcal{I}$ (note: if S_n is dense in $\mathcal{I}$ then S_n is dense in $C(I)$). Let $f \in \mathcal{I} \setminus S_n$ with $\epsilon > 0$. Since $CR : C(I) \rightarrow \mathcal{K}$ is upper semicontinuous ([4]: Proposition V.38), there exists $\delta > 0$ so that $\|f - f_1\| < \delta$ implies $CR(f_1 \circ g) \subseteq B_\epsilon(CR(f \circ g))$. Take $\delta > 0$ so that $CR(f_1 \circ g) \subseteq B_{\frac{1}{2n}}(CR(f \circ g))$ whenever $\|f - f_1\| < \delta$, and let $\{x_1, x_2, \dots, x_m\} \subseteq CR(f \circ g)$ be a $\frac{1}{2n}$ -chain of $CR(f \circ g)$. Now, choose f_1 in $C(I)$ so that $x_i \in S_0(f_1 \circ g)$ for $1 \leq i \leq m$ and $\|f - f_1\| < \min\{\delta, \epsilon\}$. We do this using Proposition 4.11. Then $CR(f \circ g) \subseteq B_{\frac{1}{2n}}(S_0(f_1 \circ g))$ since $\{x_1, x_2, \dots, x_m\} \subseteq S_0(f_1 \circ g)$, and $CR(f_1 \circ g) \subseteq B_{\frac{1}{2n}}(CR(f \circ g))$ so that $CR(f_1 \circ g) \subseteq B_{\frac{1}{n}}(S_0(f_1 \circ g))$. We conclude that $H(\overline{S_0(f_1 \circ g)}, CR(f_1 \circ g)) < \frac{1}{n}$.

(3) We now show that S_n is an open subset of $\mathcal{I}$. Let $f \in S_n$ with $n \geq 4$. Say $H(\overline{S_0(f \circ g)}, CR(f \circ g)) = \alpha < \frac{1}{n}$, and set $\gamma = \frac{1}{n} - \alpha$. Let $\delta_1 > 0$ so that $\|f - f_1\| < \delta_1$ implies $CR(f_1 \circ g) \subseteq B_{\frac{\gamma}{2}}(CR(f \circ g))$. Take $\{x_1, x_2, \dots, x_m\} \subseteq S_0(f \circ g)$ to be an $\alpha + \frac{\gamma}{n}$ -net of $CR(f \circ g)$. Now, there exists $\delta_2 > 0$ so that $\|f - f_1\| < \delta_2$ implies $S_0(f_1 \circ g) \cap B_{\frac{\gamma}{n}}(x_i) \neq \emptyset$ for $i = 1, 2, \dots, m$. If $f_1 \in \mathcal{I}$ for which $\|f - f_1\| < \min\{\delta_1, \delta_2\}$ then $\cup_{i=1}^m B_{\frac{\gamma}{n}}(x_i) \subseteq B_{\frac{\gamma}{n}}(S_0(f_1 \circ g))$, $CR(f \circ g) \subseteq B_{\alpha + \frac{\gamma}{n}}(\cup_{i=1}^m B_{\frac{\gamma}{n}}(x_i))$ and $CR(f_1 \circ g) \subseteq B_{\frac{\gamma}{n}}(CR(f \circ g))$. It follows that $CR(f_1 \circ g) \subseteq B_{\frac{1}{n}}(S_0(f_1 \circ g))$ so that $H(\overline{S_0(f_1 \circ g)}, CR(f_1 \circ g)) < \frac{1}{n}$ and $f_1 \in S_n$. $\square$

Corollary 4.14. If $g \in \mathcal{I}$ then there exists a residual set $\mathcal{S} \subseteq C(I)$ such that the map $\Lambda : C(I) \times C(I) \rightarrow \mathcal{K}$ is continuous at (f, g) whenever $f \in \mathcal{S}$.

5. THE COLLECTION OF ω -LIMIT SETS OF $[f, g]$

Let $(\mathcal{K}^*, \mathcal{H}^*)$ be the metric space consisting of all non-empty closed subsets of $\mathcal{K}$. Thus, $K \in \mathcal{K}^*$ if K is a non-empty family of non-empty closed sets in I such that K is closed in $\mathcal{K}$ with respect to $\mathcal{H}$. We endow $\mathcal{K}^*$ so that K_1 and K_2 are close with respect to $\mathcal{H}^*$ if each member of K_1 is close to some member of K_2 with respect to $\mathcal{H}$, and viceversa. This metric space is compact ([6], [23]).

Definition 5.1. Define $\mathcal{L} : C(I) \rightarrow \mathcal{K}^*$ as $f \mapsto \mathcal{L}(f) = \{\omega(x, f) : x \in I\}$.

Theorem 5.2. ([16]: Theorem 4.6) Let f be an element of $C(I)$. Then $\mathcal{L}(f)$ is contained in the Hausdorff closure of the finite ω -limit sets of f whenever $\overline{P(f)} = CR(f)$.

Theorem 5.3. Let f and g be elements of $C(I)$. Then $\mathcal{L}([f, g])$ is contained in the Hausdorff closure of the finite ω -limit sets of $[f, g]$ whenever $\overline{P(g \circ f)} = CR(g \circ f)$.

Proof. Let $\beta \in \mathcal{L}([f, g])$ with $\epsilon > 0$. We want to establish the existence of $\sigma \in \mathcal{L}([f, g])$, σ finite, so that $\mathcal{H}(\beta, \sigma) < \epsilon$. Since $\beta \in \mathcal{L}([f, g])$, there exist $A \in \mathcal{L}(g \circ f)$ and $B \in \mathcal{L}(f \circ g)$, necessarily unique, such that $\beta = A \cup B$, and $f(A) = B$. Since $f \in C(I)$, there exists $\delta > 0$, with $\epsilon > \delta$, so that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Now, consider $A \in \mathcal{L}(g \circ f)$. By hypothesis, $\overline{P(g \circ f)} = CR(g \circ f)$, so, by Theorem 5.2, there is $\alpha \in \mathcal{L}(g \circ f)$ finite, say $\alpha = \omega(x, g \circ f)$, such that $\mathcal{H}(A, \alpha) < \delta$. The continuity of f insures that $\mathcal{H}(f(A), f(\alpha)) = \mathcal{H}(B, \omega(f(x), f \circ g)) < \epsilon$. Thus, $\mathcal{H}(\beta, \sigma) < \epsilon$, where $\sigma = \alpha \cup f(\alpha) = \omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$. $\square$

Theorem 5.4. The map $\mathcal{L} : C(I) \times C(I) \rightarrow \mathcal{K}^*$ is continuous at (f, g) if the map $\mathcal{L} : C(I) \rightarrow \mathcal{K}^*$ is continuous at $g \circ f$.

Proof. Suppose that $\{(f_n, g_n)\}_{n \in \mathbb{N}} \subseteq C(I) \times C(I)$ so that $(f_n, g_n) \rightarrow (f, g)$. It suffices to show that $\mathcal{L}([f_n, g_n]) \rightarrow \mathcal{L}([f, g])$.

(1) Since $(f_n, g_n) \rightarrow (f, g)$, and $\mathcal{L}$ is continuous at $g \circ f$, it follows that $g_n \circ f_n \rightarrow g \circ f$ and $\mathcal{L}(g_n \circ f_n) \rightarrow \mathcal{L}(g \circ f)$.

(2) Since $\mathcal{L}(g_n \circ f_n) \rightarrow \mathcal{L}(g \circ f)$ and $f_n \rightarrow f$, it follows that $f_n(\mathcal{L}(g_n \circ f_n)) = \mathcal{L}(f_n \circ g_n) \rightarrow f(\mathcal{L}(g \circ f)) = \mathcal{L}(f \circ g)$.

(3) There is a bijection from $\mathcal{L}(g \circ f)$ to $\mathcal{L}([f, g])$ given by $\omega(x, g \circ f) \mapsto \omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$, so that $\omega(x, [f, g])$ can be considered as an element of $\mathcal{L}(g \circ f) \times \mathcal{L}(f \circ g)$.

We conclude that $\mathcal{L}([f_n, g_n]) \rightarrow \mathcal{L}([f, g])$. $\square$

Definition 5.5. We say that a periodic orbit A of f of period $n \geq 1$ is a p -stable periodic orbit if for any $\epsilon > 0$ there is a $\delta > 0$ such that any $g \in C(I)$ with $\|f - g\| < \delta$ has a periodic orbit B of period n satisfying $\mathcal{H}(A, B) < \epsilon$. Denote by $\mathbf{S}(f)$ the collection of p -stable periodic orbits of f . Following immediately from these definitions we have

$$x \in S_0(f) \rightarrow \omega(x, f) \in \mathbf{S}(f).$$

Definition 5.6. ([24]: page 506; [21]) Let $f \in C(I)$. Let $\tilde{\Omega}(f)$ be the collection of all subsets L of $[0, 1]$ satisfying all of these properties together:

- i. L is closed,
- ii. $f(L) = L$, and
- iii. for any nonempty proper closed $F \subset L$, $F \cap \overline{f(L \setminus F)} \neq \emptyset$.

Proposition 5.7. ([24]: Proposition 4.1) *Let $f \in C(I)$. Then, $\tilde{\Omega}(f)$ is closed in $(\mathcal{K}, \mathcal{H})$.*

Proposition 5.8. ([24]: Proposition 4.2) *The map $\tilde{\Omega} : C(I) \rightarrow \mathcal{K}^*$ given by $f \mapsto \tilde{\Omega}(f)$ is upper semicontinuous.*

Proposition 5.9. ([24]: Proposition 4.3) *If $f \in C(I)$ for which $\overline{\mathbf{S}}(f) = \tilde{\Omega}(f)$ in $(\mathcal{K}^*, \mathcal{H}^*)$, then $\mathcal{L} : C(I) \rightarrow \mathcal{K}^*$ is continuous at f .*

Corollary 5.10. *Let $(f, g) \in C(I) \times C(I)$. If $\overline{\mathbf{S}(f \circ g)} = \tilde{\Omega}(f \circ g)$, then the map $\mathcal{L} : C(I) \times C(I) \rightarrow \mathcal{K}^*$ is continuous at (f, g) .*

We want to show that, typically, the map $\mathcal{L} : C(I) \times C(I) \rightarrow \mathcal{K}^*$ given by $(f, g) \mapsto \mathcal{L}([f, g])$ is continuous. From Theorem 5.4, it suffices to show that, typically, the map taking $(f, g) \mapsto \mathcal{L}(g \circ f)$ is continuous. From Proposition 5.9 it suffices to show that $\overline{\mathbf{S}}(g \circ f) = \tilde{\Omega}(g \circ f)$ for the typical (f, g) .

Lemma 5.11. *If $L \in \tilde{\Omega}(f)$ then for any $\epsilon > 0$ there exists an ϵ -chain $\chi = \{x_0, x_1, \dots, x_n\} \subseteq L$ such that $x_0 = x_n$ and $\mathcal{H}(\chi, L) < \epsilon$.*

Proof. Since f is uniformly continuous there is $\delta > 0$ so that $|x - y| < \delta < \frac{\epsilon}{4}$ implies $|f(x) - f(y)| < \frac{\epsilon}{4}$. Since L is compact we can take $\{y_0, y_1, \dots, y_m\} \subseteq L$ so that

- (1) $y_i < y_j$ whenever $i < j$,
- (2) $L \subseteq \bigcup_{i=0}^m B_\delta(y_i)$, and $L \cap B_\delta(y_i) \neq \emptyset$, for every $0 \leq i \leq m$
- (3) $B_\delta(y_0) \cap B_\delta(y_i) \neq \emptyset \Rightarrow i = 1$, $B_\delta(y_m) \cap B_\delta(y_i) \neq \emptyset \Rightarrow i = m - 1$; if $j = 1, 2, \dots, m - 1$ then $B_\delta(y_j) \cap B_\delta(y_i) \neq \emptyset \Rightarrow i = j - 1$ or $i = j + 1$,
- (4) if $j \neq i$ then $y_j \notin B_\delta(y_i)$

Our goal is to create an ϵ -chain $\{x_0, x_1, \dots, x_n\}$ using the points $\{y_0, y_1, \dots, y_m\}$ so that $x_0 = x_n$ and for $0 \leq j \leq m$ there exists $0 \leq i \leq n$ so that $y_j = x_i$. This will insure that $\mathcal{H}(\chi, L) < \epsilon$.

First, let $x_0 = y_0$. Recall that $F \cap \overline{f(L \setminus F)} \neq \emptyset$ for any nonempty proper closed $F \subseteq L$. Let $F = \overline{\bigcup_{j=1}^m B_\delta(y_j)}$. Since $F \cap \overline{f(L \setminus F)} \neq \emptyset$, there exists $x \in B_\delta(y_0)$ so that $f(x) \in \overline{B_\delta(y_i)}$, $i \neq 0$. Thus, $|f(y_0) - y_i| < |f(y_0) - f(x)| + |f(x) - y_i| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$, so $\{y_0, y_1\}$ is an $\frac{\epsilon}{2}$ -chain.

Now, let $F = \overline{\bigcup_{l \neq i} B_\delta(y_l)}$. Since $F \cap \overline{f(L \setminus F)} \neq \emptyset$, there exists $x \in B_\delta(y_i)$ so that $f(x) \in \overline{B_\delta(y_s)}$, $s \neq i$. Thus, $|f(y_i) - y_s| < |f(y_i) - f(x)| + |f(x) - y_s| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$, so $\{y_i, y_s\}$ is an $\frac{\epsilon}{2}$ -chain.

Let $x_0 = y_0$, $x_1 = y_i$ and $x_2 = y_s$. Then, $\{x_0, x_1, x_2\}$ is an $\frac{\epsilon}{2}$ -chain.

Arguing in this way, assume that at a certain step, we have constructed an $\frac{\epsilon}{2}$ -chain $\{x_0, x_1, \dots, x_t = y_r\}$, where $t \in \mathbb{N}$, $\{x_0, x_1, \dots, x_t\} \subseteq \{y_0, y_1, \dots, y_m\}$. We can always extend $\{x_0, x_1, \dots, x_t = y_r\}$ to $\{x_0, x_1, \dots, x_t, x_{t+1}\}$ as an $\frac{\epsilon}{2}$ -chain, as there exists $x \in B_\delta(x_t)$ so that $f(x) \in \overline{B_\delta(y_h)}$, $h \neq t$. Moreover, because $\{y_0, y_1, \dots, y_m\}$ contains $m + 1$ elements, for some $h \leq m + 2$ we will arrive at an $\frac{\epsilon}{2}$ -chain $\{x_0, x_1, \dots, x_h\}$ with $x_h = x_j$ where $x_j \in \{x_0, x_1, \dots, x_{h-1}\}$. That is, $\{x_j, x_{j+1}, \dots, x_h\}$ is an $\frac{\epsilon}{2}$ -chain where the first element coincides with the last one. If $\{y_0, y_1, \dots, y_m\} \subseteq \{x_j, x_{j+1}, \dots, x_h\}$, then we are done. Suppose, then, that $\{y_0, y_1, \dots, y_m\}$ is not a subset of $\{x_j, x_{j+1}, \dots, x_h\}$. We show that when this is the case $\{x_j, x_{j+1}, \dots, x_h\}$ can always be extended as an $\frac{\epsilon}{2}$ -chain to include some $y_i \in \{y_0, y_1, \dots, y_m\} \setminus \{x_j, x_{j+1}, \dots, x_h\}$. From this our conclusion will follow. To this

end, let $F = \overline{\cup_{p \in J} B_\delta(y_p)}$, where $p \in J$ if $y_p \in \{y_0, y_1, \dots, y_m\} \setminus \{x_j, x_{j+1}, \dots, x_h\}$. Then there exists $x \in B_\delta(x_k)$, $j \leq k \leq h$, so that $f(x) \in \overline{B_\delta(y_p)}$, where $p \in J$. This allows us to form the ϵ -chain $\{x_j, \dots, x_h = x_j, x_{j+1}, \dots, x_k, y_p\}$. Say $y_p = z_1$. As before, we can extend $\{x_j, \dots, z_1\}$ to $\{x_j, \dots, z_1, z_2\}$ where $z_2 = y_j$ for some $0 \leq j \leq m$. We continue to do this till we get $s \geq 2$ minimal so that $z_s \in \{x_j, \dots, x_h\}$. Say $z_s = x_q$. Then

$$\{x_j, x_{j+1}, \dots, x_h = x_j, x_{j+1}, \dots, x_k, z_1, z_2, \dots, z_s = x_q, x_{q+1}, \dots, x_h\}$$

is an $\frac{\epsilon}{2}$ -chain whose first and last element coincide, and contains at least one more element of $\{y_0, y_1, \dots, y_m\}$ than does $\{x_j, \dots, x_n\}$ $\square$

Lemma 5.12. *Let f and g be in $\mathcal{I}$, $L \in \tilde{\Omega}(f \circ g)$ and $\epsilon > 0$. There exists f_1 in $C(I)$ and $K \in \mathcal{L}(f_1 \circ g)$ so that $\|f - f_1\| < \epsilon$, K is finite and $\mathcal{H}(K, L) < \epsilon$.*

Proof. Using Lemma 5.11, let $\beta = \{x_0, x_1, \dots, x_n\} \subseteq L$ be an $\frac{\epsilon}{2}$ -chain with respect to $f \circ g$ so that $x_0 = x_n$ and $\mathcal{H}(L, \beta) < \frac{\epsilon}{2}$. We define an appropriate $h : I \rightarrow I$ so that $\|h\| < \epsilon$, and set $f_1 = h \circ f$. The starting point for our development of h is to take $h((f \circ g)(x_i)) = x_{i+1}$, but the $\{x_i\}_{i=0}^{n-1}$ may not be distinct. Thus, we use points y_i close to x_i that get mapped to points y_{i+1} close to x_{i+1} by $f \circ g$. That we are able to find distinct points y_i follows from the fact that f and g are contained in $\mathcal{I}$.

Since $f \circ g$ is uniformly continuous, there exists $\frac{\epsilon}{4} > \delta > 0$ so that $|(f \circ g)(x) - (f \circ g)(y)| < \frac{\epsilon}{4}$ whenever $|x - y| < \delta$. We define h as follows:

- (1) Take $y_0 \in B_\delta(x_0)$ so that $y_0 \notin \beta$ and $(f \circ g)(y_0) \notin \beta$. Pick $y_1 \in B_\delta(x_1)$ so that $y_1 \notin \beta \cup \{y_0\}$ and $(f \circ g)(y_1) \notin \beta \cup \{y_0\}$. Set $h((f \circ g)(y_0)) = y_1$.
- (2) In general, for $0 < i < n-1$, take $y_i \in B_\delta(x_i)$ so that $y_i \notin \beta \cup \{y_0, y_1, \dots, y_{i-1}\}$ and $(f \circ g)(y_i) \notin \beta \cup \{y_0, y_1, \dots, y_{i-1}\}$. Set $h((f \circ g)(y_{i-1})) = y_i$.
- (3) For $i = n-1$, set $h((f \circ g)(y_{n-1})) = y_n = y_0$. Now, for any $i = 0, 1, \dots, n-1$, we have

$$\begin{aligned} & |h((f \circ g)(y_i)) - (f \circ g)(y_i)| = |y_{i+1} - (f \circ g)(y_i)| \\ &= |(y_{i+1} - x_{i+1}) + (x_{i+1} - (f \circ g)(x_i)) + ((f \circ g)(x_i) - (f \circ g)(y_i))| \\ &\leq |y_{i+1} - x_{i+1}| + |x_{i+1} - (f \circ g)(x_i)| + |(f \circ g)(x_i) - (f \circ g)(y_i)| \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

- (4) Now, extend h to all of $[0, 1]$ so that $\|h\| < \epsilon$ using the Tietze extension theorem.

It follows that $\|h \circ f - f\| < \epsilon$. Moreover, $\omega(y_0, (h \circ f) \circ g) = \cup_{i=0}^{n-1} y_i$ is an n -cycle and $\mathcal{H}(\omega(y_0, (h \circ f) \circ g), \beta) < \delta$. Since $B_\delta(y_i) \cap \beta \neq \emptyset$ for any i , one has $\mathcal{H}(\omega(y_0, (h \circ f) \circ g), \beta) < \delta$. Since $\mathcal{H}(\beta, L) < \frac{\epsilon}{2}$ we conclude that $\mathcal{H}(\omega(y_0, (h \circ f) \circ g), L) < \delta + \frac{\epsilon}{2} < \epsilon$. Set $h \circ f = f_1$, and $\omega(y_0, (h \circ f) \circ g) = \omega(y_0, f_1 \circ g) = K$. $\square$

From Proposition 4.11, we have the following corollary to Lemma 5.12.

Corollary 5.13. *Let f and g be in $\mathcal{I}$, $L \in \tilde{\Omega}(f \circ g)$ and $\epsilon > 0$. Then there exists f_1 in $C(I)$ and $K \in \mathbf{S}(f_1 \circ g)$ so that $\|f - f_1\| < \epsilon$ and $\mathcal{H}(K, L) < \epsilon$.*

Theorem 5.14. *Let $g \in \mathcal{I}$. The set $W = \{f \in C(I) : \overline{\mathbf{S}(f \circ g)} = \tilde{\Omega}(f \circ g)\}$ is residual in $(C(I), \|\cdot\|)$.*

Proof. Let $B_n = \{f \in \mathcal{I} : \mathcal{H}^*(\overline{\mathbf{S}(f \circ g)}, \tilde{\Omega}(f \circ g)) > \frac{1}{n}\}$. It suffices to show that B_n is nowhere dense for any n .

We first show that $\mathcal{I} \setminus B_n$ is dense. Let $f \in B_n$. Since $\tilde{\Omega} : C(I) \rightarrow \mathcal{K}^*$ is upper semicontinuous, there exists $\delta > 0$ so that $\tilde{\Omega}(h \circ g) \subseteq B_{\frac{1}{4n}}(\tilde{\Omega}(f \circ g))$ whenever $d(h, f) < \delta$. Since $\tilde{\Omega}(f \circ g)$ is closed in $\mathcal{K}$, there exist $\{L_i\}_{i=1}^m \subseteq \tilde{\Omega}(f \circ g)$ so that $\{L_i\}_{i=1}^m$ is a $\frac{1}{4n}$ -net of $\tilde{\Omega}(f \circ g)$; that is, for any $K \in \tilde{\Omega}(f \circ g)$ there exists i so that $1 \leq i \leq m$ and $\mathcal{H}(K, L_i) < \frac{1}{4n}$. Choose $f_1 \in C(I)$ so that $\|f_1 - f\| < \delta$ and there is a stable periodic orbit $K_i \in \mathbf{S}(f_1 \circ g)$ so that $\mathcal{H}(K_i, L_i) < \frac{1}{4n}$ for $1 \leq i \leq m$. It follows that $\tilde{\Omega}(f_1 \circ g) \subseteq B_{\frac{1}{4n}}(\tilde{\Omega}(f \circ g)) \subseteq B_{\frac{1}{2n}}(\{L_i\}_{i=1}^m) \subseteq B_{\frac{3}{4n}}(\overline{\mathbf{S}(f_1 \circ g)})$, so that $\mathcal{H}^*(\overline{\mathbf{S}(f_1 \circ g)}, \tilde{\Omega}(f_1 \circ g)) < \frac{1}{n}$.

We now show that $\mathcal{I} \setminus B_n$ is open. Let $f \in \mathcal{I}$ such that $\mathcal{H}^*(\mathbf{S}(f \circ g), \tilde{\Omega}(f \circ g)) = \sigma < \frac{1}{n}$, say $\frac{1}{n} - \sigma = \epsilon$. Choose $\delta_1 > 0$ so that $\tilde{\Omega}(f_1 \circ g) \subseteq B_{\frac{\epsilon}{4}}(\tilde{\Omega}(f \circ g))$ whenever $\|f_1, f\| < \delta_1$, and take $\{L_i\}_{i=1}^m \subseteq \mathbf{S}(f \circ g)$ with the property that $\mathcal{H}^*(\{L_i\}_{i=1}^m, \tilde{\Omega}(f \circ g)) < \sigma + \frac{\epsilon}{4}$. Since $\{L_i\}_{i=1}^m \subseteq \mathbf{S}(f \circ g)$ there exists $\delta_2 > 0$ so that $\|f_1 - f\| < \delta_2$ implies the existence, for any $1 \leq i \leq m$, of $K_i \in \mathbf{S}(f_1 \circ g)$ so that $\mathcal{H}(K_i, L_i) < \frac{\epsilon}{4}$. Let $f_1 \in \mathcal{I}$ with $\|f_1 - f\| < \min\{\delta_1, \delta_2\}$. Then $\tilde{\Omega}(f_1 \circ g) \subseteq B_{\frac{\epsilon}{4}}(\tilde{\Omega}(f \circ g)) \subseteq B_{\sigma + \frac{\epsilon}{2}}(\{L_i\}_{i=1}^m) \subseteq B_{\sigma + \frac{3\epsilon}{4}}(\{K_i\}_{i=1}^m) \subseteq B_{\sigma + \frac{3\epsilon}{4}}(\overline{\mathbf{S}(f_1 \circ g)})$, so that $\mathcal{H}^*(\overline{\mathbf{S}(f_1 \circ g)}, \tilde{\Omega}(f_1 \circ g)) < \sigma + \frac{3\epsilon}{4} < \frac{1}{n}$. $\square$

Corollary 5.15. *If $g \in \mathcal{I}$ then there exists a residual set $\mathcal{S} \subseteq C(I)$ so that the map $\mathcal{L} : C(I) \times C(I) \rightarrow \mathcal{K}^*$ is continuous at (f, g) whenever $f \in \mathcal{S}$.*

6. THE MAP ω

Theorem 6.1. *The map $\omega : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ given by $(x, f, g) \mapsto \omega(x, [f, g])$ is continuous at (x_0, f_0, g_0) if the map $\omega : I \times C(I) \rightarrow \mathcal{K}$ given by $(x, f) \mapsto \omega(x, f)$ is continuous at $(x_0, g_0 \circ f_0)$.*

Proof. Let $\{(x_n, f_n, g_n)\}_{n=1}^\infty \subseteq I \times C(I) \times C(I)$ such that $(x_n, f_n, g_n) \rightarrow (x_0, f_0, g_0)$. Since $(f_n, g_n) \rightarrow (f_0, g_0)$ in $C(I) \times C(I)$, one sees that $g_n \circ f_n \rightarrow g_0 \circ f_0$ in $C(I)$. Thus, $(x_n, g_n \circ f_n) \rightarrow (x_0, g_0 \circ f_0)$, and since $\omega : I \times C(I) \rightarrow \mathcal{K}$ is continuous at $(x_0, g_0 \circ f_0)$, it follows that $\omega(x_n, g_n \circ f_n) \rightarrow \omega(x_0, g_0 \circ f_0)$. Now, recall that $\omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$, with $\omega(f(x), f \circ g) = f(\omega(x, g \circ f))$. Since $f_n \rightarrow f_0$ in $C(I)$ and $\omega(x_n, g_n \circ f_n) \rightarrow \omega(x_0, g_0 \circ f_0)$, it follows that

$$f_n(\omega(x_n, g_n \circ f_n)) = \omega(f_n(x_n), g_n \circ f_n) \rightarrow \omega(f_0(x_0), f_0 \circ g_0).$$

We conclude that

$$\begin{aligned} \omega(x_n, [f_n, g_n]) &= \omega(x_n, g_n \circ f_n) \cup \omega(f_n(x_n), f_n \circ g_n) \\ &\rightarrow \omega(x_0, [f_0, g_0]) = \omega(x_0, g_0 \circ f_0) \cup \omega(f_0(x_0), f_0 \circ g_0). \end{aligned}$$

$\square$

Theorem 6.2. *Let $g \in C(I)$. There exists a residual set $\mathcal{T} \subseteq I \times C(I)$ so that the map $\omega : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ is continuous at (x, g, f) whenever $(x, f) \in \mathcal{T}$.*

Proof. From Theorem 6.1 it suffices to find a residual subset $\mathcal{T} \subseteq I \times C(I)$ so that $(x_0, f_0) \in \mathcal{T}$ implies that the map $\omega : I \times C(I) \rightarrow \mathcal{K}$ given by $(x, f) \mapsto \omega(x, f \circ g)$ is continuous at (x_0, f_0) . Moreover, since the set of points of continuity for any map is a G_δ set, one need only show that ω is continuous on a dense set of points in $I \times C(I)$.

Fix $g \in \mathcal{C}(I)$. Let $(x, f) \in I \times C(I)$ and take $\epsilon > 0$. Without loss of generality, we may assume that $\omega(x, f \circ g)$ is infinite. Take $z \in \omega(x, f \circ g)$ and n_1, n_2 in $\mathbb{N}$, so that $|(f \circ g)^{n_1}(x) - z| < \frac{\epsilon}{4}$ and $|(f \circ g)^{n_2}(x) - z| < \frac{\epsilon}{4}$. We want f_1 close to f so that $(f_1 \circ g)^{n_2}(x) = (f_1 \circ g)^{n_1}(x) = (f \circ g)^{n_1}(x)$. Consider the set $\{g((f \circ g)^j(x)) : 0 \leq j \leq n_2\}$. These points are distinct since $\omega(x, f \circ g)$ is infinite. Let $\delta > 0$ so that $B_\delta(g((f \circ g)^{n_2-1}(x))) \cap g((f \circ g)^j(x)) = \emptyset, 0 \leq j < n_2 - 1$. Define f_1 so that $f_1(z) = f(z)$ whenever $z \in I \setminus B_\delta(g((f \circ g)^{n_2-1}(x)))$, $f_1(g((f \circ g)^{n_2-1}(x))) = (f \circ g)^{n_1}(x)$, and use the Tietze extension theorem to get $f_1 \in C(I)$ so that $\|f_1 - f\| < \frac{\epsilon}{2}$. This is possible since

$$\begin{aligned} & |f_1(g((f \circ g)^{n_2-1}(x))) - (f \circ g)^{n_2}(x)| \\ &= |f_1(g((f \circ g)^{n_2-1}(x))) - f(g((f \circ g)^{n_2-1}(x)))| \\ &= |(f \circ g)^{n_1}(x) - (f \circ g)^{n_2}(x)| \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Now, consider the trajectory $\gamma(x, f_1 \circ g) = \{x, (f_1 \circ g)(x), (f_1 \circ g)^2(x), \dots, (f_1 \circ g)^{n_1}(x), \dots, (f_1 \circ g)^{n_2}(x) = (f_1 \circ g)^{n_1}(x), \dots\}$ which gives us the periodic orbit $\{(f_1 \circ g)^{n_1}(x), (f_1 \circ g)^{n_1+1}(x), \dots, (f_1 \circ g)^{n_2-1}(x)\}$.

Choose $\delta > 0$ so that $B_\delta(g((f_1 \circ g)^{n_2-1}(x))) \cap g((f_1 \circ g)^j(x)) = \emptyset, n_1 < j < n_2 - 2$ and $s, t \in I$ with $|s - t| < \delta$ implies $|f_1(t) - f_1(s)| < \frac{\epsilon}{2}$. Define f_2 so that $f_2(z) = f_1(z)$ whenever $z \in I \setminus B_\delta(g(f_1 \circ g)^{n_2-1}(x))$, $f_2(z) = f_1(g(f_1 \circ g)^{n_2-1}(x))$ whenever $z \in B_{\frac{\delta}{2}}(g(f_1 \circ g)^{n_2-1}(x))$ and use the Tietze extension theorem to get $f_2 \in C(I)$ so that $\|f_1 - f_2\| < \frac{\epsilon}{2}$. This is possible since $|f_2(z) - f_1(z)| < \frac{\epsilon}{2}$ for all $z \in B_{\frac{\delta}{2}}(g(f_1 \circ g)^{n_2-1}(x))$. Now, $f_2 \circ g$ is constant on some interval which contains $(f_2 \circ g)^{n_2-1}(x)$ in its interior. By Proposition 3.1 in [25], it follows that $\omega : I \times C(I) \rightarrow \mathcal{K}$ is continuous at $(x, f_2 \circ g)$.

□

We omit the proof of the following lemma as it is a standard induction calculus argument.

Lemma 6.3. *The set $\{(x, f, g) : |(f \circ g)^k(x) - a| < \epsilon\}$ is open in $I \times C(I) \times C(I)$ for all natural numbers k and all a in I .*

Proposition 6.4. *The map $\omega_1 : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ given by $(x, f, g) \mapsto \omega(x, g \circ f)$ is in the second class of Baire.*

Proof. Let L be a compact set in I and $\{a_i : i = 1, 2, 3, \dots\}$ be a countable dense subset of L . Let $\epsilon > 0$. It suffices to show that $\{(x, f, g) : \mathcal{H}(\omega(x, g \circ f), L) < \epsilon\}$ is a $G_{\delta\sigma}$ subset of $I \times C(I) \times C(I)$. Set $A = \{(x, f, g) : \omega(x, g \circ f) \subseteq B_\epsilon(L)\}$ and $B = \{(x, f, g) : L \subseteq B_\epsilon(\omega(x, g \circ f))\}$. Note that $\mathcal{H}(\omega(x, g \circ f), L) < \epsilon$ if and only if $(x, f, g) \in A \cap B$. One verifies easily that

$$A = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{(x, f, g) : d((g \circ f)^k(x), L) \leq \frac{n\epsilon}{n+1}\}$$

and

$$B = \bigcup_{n=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{(x, f, g) : |(g \circ f)^k(x) - a_j| < \frac{n\epsilon}{n+1}\}.$$

Since A is an F_σ set and B is a $G_{\delta\sigma}$ set it follows that $A \cap B$ is a $G_{\delta\sigma}$ set.

□

Corollary 6.5. *The map $\omega : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ given by $(x, f, g) \mapsto \omega(x, [f, g])$ is in the second class of Baire.*

Proof. Let $\omega_2 : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ be given by $(x, f, g) \mapsto \omega(f(x), f \circ g)$. Since $\omega_2 = f \circ \omega_1$, one sees that ω_2 is the composition of a continuous function with a Baire 2 function, so that ω_2 is in the second class of Baire, too. Our conclusion follows by recalling that $\omega : I \times C(I) \times C(I) \rightarrow \mathcal{K}$ is given by $(x, f, g) \mapsto \omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g) = \omega_1(x, f, g) \cup (f \circ \omega_1)(x, f, g)$. $\square$

7. TYPICAL ω -LIMIT SETS OF $[f, g]$

Lemma 7.1. *Let $x \in I$, $f, g \in \mathcal{I}$ and $\epsilon > 0$ and denote by $\psi = g \circ f$. Then there exists $h \in \mathcal{C}(I)$ such that $h \in B_\epsilon(i_d)$ and $\text{orb}(x, h \circ \psi)$ is finite.*

Proof. Of course, if $\text{orb}(x, \psi)$ is finite, we choose $h = i_d$. So, assume that $\text{orb}(x, \psi)$ is infinite, and let p be a limit point of $\text{orb}(x, \psi)$. Let j be the least positive integer such that $\psi^j(x) \in B_{\frac{\epsilon}{2}}(p)$ and $k > 0$ the least positive integer such that $\psi^{j+k}(x) \in B_{\frac{\epsilon}{2}}(p)$. Now, $\eta = \min\{|\psi^m(x) - \psi^n(x)| : 0 \leq m, n \leq j+k, m \neq n\}$ and $\sigma = \min\{\epsilon, \eta\}$. Define $h : I \rightarrow I$ so that

- (1) $h(\psi^{j+k}(x)) = \psi^j(x)$
- (2) $h = i_d$ on $I \setminus B_{\frac{\sigma}{2}}(\psi^{j+k}(x))$
- (3) use the Tietze extension theorem to extend h to all of I such that $\|h - i_d\| < \epsilon$.

$\square$

Lemma 7.2. *Let $x \in I$, $f, g \in \mathcal{I}$, $t \in \mathbb{N}$ and $\epsilon > 0$ and let $\psi = g \circ f$. If $\text{orb}(x, \psi)$ is finite then there exists $h \in \mathcal{C}(I)$ such that $h \in B_\epsilon(i_d)$, $\text{orb}(x, h \circ \psi)$ is finite and $|\omega(x, h \circ \psi)|$ is a multiple of t .*

Proof. Let $\omega(x, \psi) = \{x_0, x_1, \dots, x_m\}$ such that $\psi(x_i) = x_{i+1}$ for $0 \leq i \leq m-1$, $\psi(x_m) = x_0$, $\psi^p(x) = x_0$, and $\psi^k(x) \neq x_i$ for any $0 \leq i \leq m$ whenever $k < p$. Let $\sigma = \min\{|y - z| : y, z \in \text{orb}(x, \psi), y \neq z\}$. Since ψ is uniformly continuous on I , there exists $\delta > 0$ so that $\text{diam}(\psi(L)) < \min\{\frac{\epsilon}{2}, \frac{\sigma}{4}\}$ whenever $\text{diam}(L) < \delta$. Set $r = \min\{\frac{\sigma}{4}, \frac{\delta}{2}, \frac{\epsilon}{2}\}$. Then,

- (1) take, $y_0^i, y_1^i, \dots, y_{t-1}^i$ in $B_r(x_i)$ such that $\psi(y_l^i) \neq \psi(y_k^i)$ whenever $l \neq k$, and $\psi(y_k^i) \neq \psi(x_i)$ for any $0 \leq k \leq t-1$,
- (2) define $h : I \rightarrow I$ so that
 - a. $h(\psi(y_j^i)) = y_j^{i+1}$ whenever $0 \leq i \leq m-1$, and for all j ,
 - b. $h(\psi(y_j^m)) = y_{j+1}^0$ when $0 \leq j \leq t-2$,
 - c. $h(\psi(y_{t-1}^m)) = y_0^0$,
 - d. $h(\psi^k(x)) = \psi^k(x)$ when $0 \leq k \leq p-1$, and
 - e. $h = i_d$ on $I \setminus \bigcup_{k=0}^{p+m} B_r(\psi^k(x))$
- (3) use the Tietze extension theorem to extend h to all of I such that $\|h - i_d\| < \epsilon$.

$\square$

Lemma 7.3. *Let $x \in I$, $f, g \in \mathcal{C}(I)$, let $\psi = g \circ f$ and assume that $x_0, x_1, \dots, x_m$ are distinct points such that $\psi(x_i) = x_{i+1}$ for all $0 \leq i \leq m-1$ and $\psi(x_m) = x_l$ for some $l \leq m$. Then, for every $\epsilon > 0$ there exist $h \in \mathcal{C}(I)$ and pairwise disjoint closed intervals $J_0, J_1, \dots, J_m$ such that*

- (1) $h \in B_\epsilon(i_d)$,
- (2) $x_i \in \text{int}(J_i)$ and $\text{diam}(J_i) < \epsilon$ for $0 \leq i \leq m$
- (3) $(h \circ \psi)(x_i) = \psi(x_i)$, for $0 \leq i \leq m$,
- (4) $(h \circ \psi)(J_i) \subseteq \text{int}(J_{i+1})$ for $0 \leq i \leq m-1$ and $(h \circ \psi)(J_m) \subseteq \text{int}(J_l)$.

Proof. Let $\sigma = \min\{|x_i - x_j| : i \neq j\}$. Since ψ is uniformly continuous on I , there exists $\delta > 0$ so that $\text{diam}(\psi(L)) < \min\{\frac{\epsilon}{2}, \frac{\sigma}{4}\}$ whenever $\text{diam}(L) < \delta$. Set $\gamma = \min\{\frac{\sigma}{4}, \frac{\delta}{2}, \frac{\epsilon}{2}\}$ and take $J_i = \overline{B_{\frac{\gamma}{2}}(x_i)}$ for each i . Define $h : I \rightarrow I$ so that

- (1) $(h(\psi(J_i)) = \psi(x_i)$ for $0 \leq i \leq m$,
- (2) $h = i_d$ on $I \setminus \cup_{i=0}^m B_\gamma(x_i)$, and
- (3) use the Tietze extension theorem to extend h to all of I such that $\|h - i_d\| < \epsilon$.

□

Corollary 7.4. *Suppose $x \in I$ and f and g are contained in $\mathcal{C}(I)$. Let $\psi = g \circ f$ and $x = x_0, x_1, \dots, x_m$ be distinct points such that $\psi(x_i) = x_{i+1}$, for all $0 \leq i \leq m-1$ and $\psi(x_m) = x_l$ for some $l \leq m$. If $\epsilon > 0$ then there exists $\eta > 0$, $g_1 \in \mathcal{C}(I)$ and pairwise disjoint compact intervals $J_0, J_1, \dots, J_m$ such that*

- (1) $g_1 \in B_\epsilon(g)$,
- (2) $x_i \in \text{int}(J_i)$ and $\text{diam}(J_i) < \epsilon$ for $0 \leq i \leq m$,
- (3) $(g_1 \circ f)(x_i) = \psi(x_i)$ for $0 \leq i \leq m$
- (4) if $g_2 \in B_\eta(g_1)$ then $(g_2 \circ f)(J_i) \subseteq \text{int}(J_{i+1})$ for $0 \leq i \leq m-1$ and $(g_2 \circ f)(J_m) \subseteq \text{int}(J_l)$.

Proof. The idea behind our proof is to set $g_1 = h \circ g$ and take $\eta = \frac{\gamma}{2}$ in accordance with Lemma 7.3.

- (1) Denote by i_d the identity map. Since $h \in B_\epsilon(i_d)$ implies $|h(y) - y| < \epsilon$ for every y , and $|(h \circ g)(x) - g(x)| = |h(y) - y|$ should $g(x) = y$, it follows that $\|h \circ g - g\| < \epsilon$.
- (2) By construction.
- (3) By construction.
- (4) We show that if $g_2 \in B_\eta(g_1)$, then $(g_2 \circ f)(J_i) \subseteq \text{int}(J_{i+1})$ for $0 \leq i \leq m-1$ and $(g_2 \circ f)(J_m) \subseteq \text{int}(J_l)$.

By construction, $(g_1 \circ f)(J_i) = x_{i+1} \in \text{int}(J_{i+1}) = B_{\frac{\gamma}{2}}(x_{i+1})$ so that $|g(f(y)) - x_{i+1}| = 0$ for every $y \in J_i$. Since $g_2 \in B_\eta(g_1)$, it follows that $|g_2(z) - g_1(z)| < \eta$ for all z in I . Now, if $y \in J_i$, then $|g_2(f(y)) - x_{i+1}| \leq |g_2(f(y)) - g_1(f(y))| + |g_1(f(y)) - x_{i+1}| < \eta + 0$, so that $(g_2 \circ f)(J_i) \subseteq B_\eta(x_{i+1}) = B_{\frac{\gamma}{2}}(x_{i+1}) = \text{int}(J_{i+1})$. Similarly, $(g_2 \circ f)(J_m) \subseteq J_l$. □

Theorem 7.5. *Let $f \in \mathcal{I}$. The set $\mathcal{A}(f) = \{(x, g) \in I \times \mathcal{C}(I) : \omega(x, g \circ f) \text{ is an odometer of type } \infty\}$ is residual in $I \times \mathcal{C}(I)$.*

Proof. Let $f \in \mathcal{I}$. Let $\{(x_i, g_i)\}$ be a dense subset of $I \times \mathcal{C}(I)$. Using our earlier lemmas, for each function g_i and $j \in \mathbb{N}$ we associate the following: a function $g_{i,j}$ such that $d(g_i, g_{i,j}) < \frac{1}{2^{i+j}}$, $0 < \eta_{i,j} < \frac{1}{2^{i+j}}$, pairwise disjoint nondegenerate compact intervals $U_{i,j}^0, U_{i,j}^1, \dots, U_{i,j}^{m_{i,j}}$ and $0 \leq l_{i,j} \leq m_{i,j}$ such that the following happen:

- (1) $|\text{orb}(x_i, g_{i,j} \circ f)| = m_{i,j} + 1 < \infty$,
- (2) $j!$ divides $m_{i,j} + 1 - l_{i,j}$,

- (3) $\text{diam}(U_{i,j}^k) < \frac{1}{2^{i+j}}$ for $0 \leq k \leq m_{i,j}$,
- (4) If $h \in B_{m_{i,j}}(g_{i,j})$ then
 - a. $(h \circ f)^k(x_i) \in (U_{i,j}^k)^o$,
 - b. $(h \circ f)(U_{i,j}^k) \subseteq (U_{i,j}^{k+1})^o$ for $0 \leq k \leq m_{i,j} - 1$,
 - c. $(h \circ f)(U_{i,j}^{m_{i,j}}) \subseteq (U_{i,j}^{l_{i,j}})^o$.

For every $j \geq l$, set $\mathcal{D}_j = \cup_{i=1}^{\infty} (U_{i,j}^0)^o \times B_{n_{i,j}}(g_{i,j})$ with $\mathcal{D} = \cap_{j=1}^{\infty} \mathcal{D}_j$. By construction, each $\mathcal{D}_j$ is a dense and open subset of $I \times \mathcal{C}(I)$. Hence, $\mathcal{D}$ is a dense G_δ subset of $I \times \mathcal{C}(I)$. We claim that for every $(y, h) \in \mathcal{D}$, $(\omega(y, h \circ f), h \circ f)$ is topologically conjugate to an odometer of type ∞ . Fix $j \in \mathbb{N}$. Since $(y, h) \in \mathcal{D}_j$, there exist $i = i(j)$, $g_{i,j}$, $m_{i,j}$, $l_{i,j}$ and $U_{i,j}^0, U_{i,j}^1, \dots, U_{i,j}^{l_{i,j}}, \dots, U_{i,j}^{m_{i,j}}$ which satisfy the above conditions (1) – (4). Let $P_j = \{U_{i,j}^{l_{i,j}}, \dots, U_{i,j}^{m_{i,j}}\}$. By condition (4), we have that $\omega(y, h \circ f) \subseteq \cup P_j$. By taking an appropriate subsequence $\{j_k\}$ we have that $P_{j_{k+1}}$ is a refinement of P_{j_k} . From conditions (2) and (3), as well as Theorem 2.11, we have that $(\omega(y, h \circ f), h \circ f)$ is topologically conjugate to an odometer of type ∞ . $\square$

Proposition 7.6. *Let f and g be in $\mathcal{C}(I)$. Then $(g \circ f, \omega(x, g \circ f))$ is α -adic if and only if $(f \circ g, \omega(f(x), f \circ g))$ is α -adic.*

Proof. We rely on Theorem 2.11 which characterizes α -adic adding machines. Suppose $(g \circ f, \omega(x, g \circ f))$ is an α -adic adding machine. Let $W_1, W_2, \dots, W_{m_k}$ be the k^{th} partition of $\omega(x, g \circ f)$, P_k , with $m_k = \alpha_1 \alpha_2 \dots \alpha_k$ such that $(g \circ f)(W_i) = W_{i+1}$ for $1 \leq i \leq m_k - 1$ and $(g \circ f)(W_{m_k}) = W_1$.

- (1) Since W_i is clopen for any i , it follows that W_i is compact. Since f is continuous, each $f(W_i)$ is compact.
- (2) Since $W_i \cap W_j = \emptyset$ whenever $i \neq j$, it follows that $f(W_i) \cap f(W_j) = \emptyset$ whenever $i \neq j$, since g is continuous.
- (3) $\{f(P_i) : 1 \leq i \leq m_k\}$ is a disjoint compact cover of $\omega(f(x), f \circ g)$, so $\{f(P_i) : 1 \leq i \leq m_k\}$ is a clopen partition of $\omega(f(x), f \circ g)$.
- (4) $(f \circ g)(f(P_i)) = f(P_{i+1})$ for $1 \leq i \leq m_k - 1$, and $(f \circ g)(f(P_{m_k})) = f(P_1)$.

If $W_1 \supset W_2 \supset W_3 \supset \dots$ is a nested sequence with $W_i \in P_i$ for each i , then $\cap_{i=1}^{\infty} W_i$ consists of a single point. Since f is continuous, if $f(W_1) \supset f(W_2) \supset f(W_3) \supset \dots$ is a nested sequence with $f(W_i) \in f(P_i)$ for each i , then $\cap_{i=1}^{\infty} f(W_i)$ consists of a single point. $\square$

Corollary 7.7. *Let $f \in \mathcal{I}$. Then there exists a residual set $\mathcal{D} \subseteq I \times \mathcal{C}(I)$ so that $\omega(x, [f, g]) = \omega(x, g \circ f) \cup \omega(f(x), f \circ g)$ where both $\omega(x, g \circ f)$ and $\omega(f(x), f \circ g)$ are adding machines of type ∞ , whenever $(x, g) \in \mathcal{D}$.*

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